

# Dynamics of large-scale quantities in Rayleigh-Bénard convection

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In this paper we estimate the relative strengths of various terms of the Rayleigh-Bénard equations. Based on these estimates and scaling analysis, we derive a general formula for the large-scale velocity,  $U$ , or the Péclet number that is applicable for arbitrary Rayleigh number  $Ra$  and Prandtl number  $Pr$ . Our formula fits reasonably well with the earlier simulation and experimental results. Our analysis also shows that the wall-bounded convection has enhanced viscous force compared to free turbulence. We also demonstrate how correlations deviate the Nusselt number scaling from the theoretical prediction of  $Ra^{1/2}$  to the experimentally observed scaling of nearly  $Ra^{0.3}$ .

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## I. INTRODUCTION

Modelling the large-scale quantities in a turbulent flow is very important in many applications, e.g., fluid and magnetohydrodynamic turbulence, Rayleigh Bénard convection (RBC), rotating turbulence, etc [1–5]. One such quantity is the magnitude of the large-scale velocity that plays a critical role in physical processes. In this paper we will quantify this and other related quantities for RBC.

RBC is an idealized version of thermal convection in which a thin layer of fluid confined between two horizontal plates separated by a distance  $d$  is heated from below and cooled from top. The properties of RBC is specified using two nondimensional parameters: the Rayleigh number  $Ra$ , which is the ratio of the buoyancy term and the diffusion term, and the Prandtl number  $Pr$ , which is the ratio of kinematic viscosity  $\nu$  and thermal diffusivity  $\kappa$  [6–9]. Some of the important quantities of interest in RBC are the large-scale velocity  $U$  or Péclet number  $Pe = Ud/\kappa$ , and the Nusselt number  $Nu$ , which is the ratio of the total heat flux to the conductive heat flux. Note that the Reynolds number  $Re = Pe/Pr$ .

The Péclet and Nusselt numbers are strong functions of  $Pr$ , as shown by Grossmann and Lohse [10–13] (referred to as GL). According to GL scaling, for small and moderate  $Pr$ 's,  $Pe \sim \sqrt{RaPr}$ , but for large  $Pr$ ,  $Pe \sim Ra^{3/5}$ . The above scaling have been verified in many experiments [14–21] and numerical simulations [22–30]. In this paper we derive a general formula for  $Pe$ , of which the aforementioned relations are limiting cases, by comparing the relative strengths of the nonlinear, pressure, buoyancy, and viscous terms, and quantifying them using the numerical data. Our derivation of the  $Pe$  formula differs from the Grossmann and Lohse's [10–13] formalism; comparison between the two approaches will be discussed towards the end of the paper.

Before describing the Péclet number formula, we

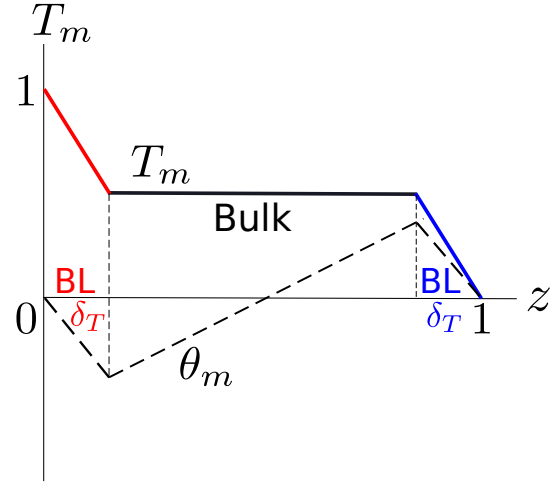


FIG. 1. The plane-averaged temperature profile  $T_m(z)$  exhibits a near constant value in the bulk, and steep variations in the thermal boundary layers whose thickness is  $\delta_T$ . The normalized temperature fluctuation  $\theta = T + z - 1$ ; the figure also exhibits its plane-averaged profile  $\theta_m(z)$ .

briefly discuss the properties of the temperature fluctuations. We normalize the temperature by the temperature difference between the plates  $\Delta$ , and the vertical coordinate  $z$  by the plate distance  $d$ . The profile of the plane-averaged temperature,  $T_m(z)$ , shown in Fig. 1, exhibits an approximate constant value of  $1/2$  in the bulk and steep variations in the thermal boundary layer. It is customary to write RBC equations in terms of the temperature fluctuation from the conduction state,  $\theta(x, y, z)$ , where  $T = T_c(z) + \theta(x, y, z)$  with  $T_c = 1 - z$  as the temperature profile for the conduction state. The planar mean of  $\theta$ ,  $\theta_m(z)$ , is illustrated in Fig. 1.

The momentum equation for RBC is

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \sigma + \alpha g \theta \hat{z} + \nu \nabla^2 \mathbf{u}, \quad (1)$$

where  $\mathbf{u}$  and  $\sigma$  are the velocity and pressure fields respectively,  $\alpha$  is the thermal expansion coefficient, and  $g$  is the acceleration due to gravity. Here we have taken the density to be unity. Under a steady state ( $\langle \partial_t \mathbf{u} \rangle = 0$ ), the acceleration of a fluid parcel,  $\mathbf{u} \cdot \nabla \mathbf{u}$ , is provided by

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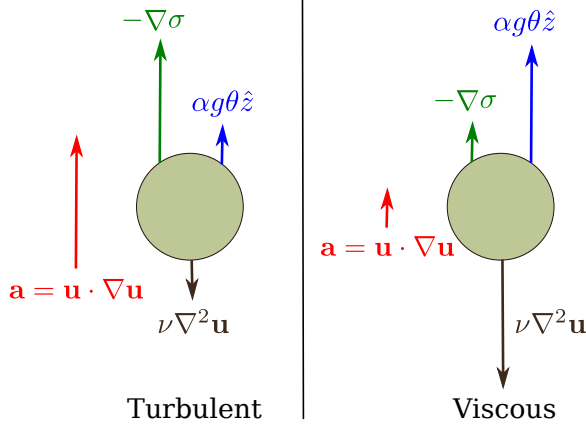


FIG. 2. Schematic depiction of acceleration  $\mathbf{a}$  of a fluid parcel that is rising against gravity: (a) in the turbulent regime,  $\mathbf{a}$  is provided primarily by the pressure gradient  $-\nabla\sigma$ ; (b) in the viscous regime,  $\mathbf{a} \approx 0$  with the buoyancy and viscous terms balancing each other. The direction of the forces are reversed for a descending fluid parcel. Note that buoyancy acts downward for the fluid parcels colder than the background.

the pressure gradient  $-\nabla\sigma$ , buoyancy  $\alpha g\theta\hat{z}$ , and the viscous term  $\nu\nabla^2\mathbf{u}$ . For a viscous flow, the net force or acceleration is very small, and the viscous force balances the buoyancy. However, in the turbulent regime, the pressure gradient dominates the buoyancy and viscous forces (see Fig. 2). We quantify these accelerations using numerical data that help us understand the scaling of the Péclet number and related quantities for arbitrary Pr and Ra. The velocity Fourier modes  $\hat{\mathbf{u}}(0,0,k_z) = 0$  due to incompressibility and no-mean flow condition. Hence  $\hat{\theta}_m(0,0,k_z)$  corresponding to  $\theta_m(z)$  follows a relation:  $ik_z\hat{\sigma}_m(0,0,k_z) = \alpha g\hat{\theta}_m(0,0,k_z)$  or  $d\sigma_m(z)/dz = \alpha g\theta_m$ . Therefore, in the momentum equation, the Fourier modes other than  $(0,0,k_z)$  involve  $\theta_{\text{res}} = \theta - \theta_m$  and  $\sigma_{\text{res}} = \sigma - \sigma_m$  respectively (see Appendix).

## II. NUMERICAL METHOD

We solve the RBC equations [6–9] in a three-dimensional unit box for  $\text{Pr} = 1, 6.8, 10^2, 10^3$  and Rayleigh numbers between  $10^6$  and  $5 \times 10^8$  on grids  $60^3, 80^3, 100^3$  and  $256^3$  using the finite volume code OpenFOAM [31]. We employ the no-slip boundary condition for the velocity field at all the walls, the conducting boundary condition at the top and bottom walls, and the insulating boundary condition at the vertical walls. For time stepping we use second-order Crank-Nicolson scheme. To resolve the boundary layers we employ a nonuniform mesh with higher concentration of grid points (greater than 4–6) near the boundaries [32, 33]. Table I includes the summary of numerical simulations performed. We have performed the grid-independence test for  $\text{Pr} = 1, \text{Ra} = 10^8$  by performing simulations on  $100^3$

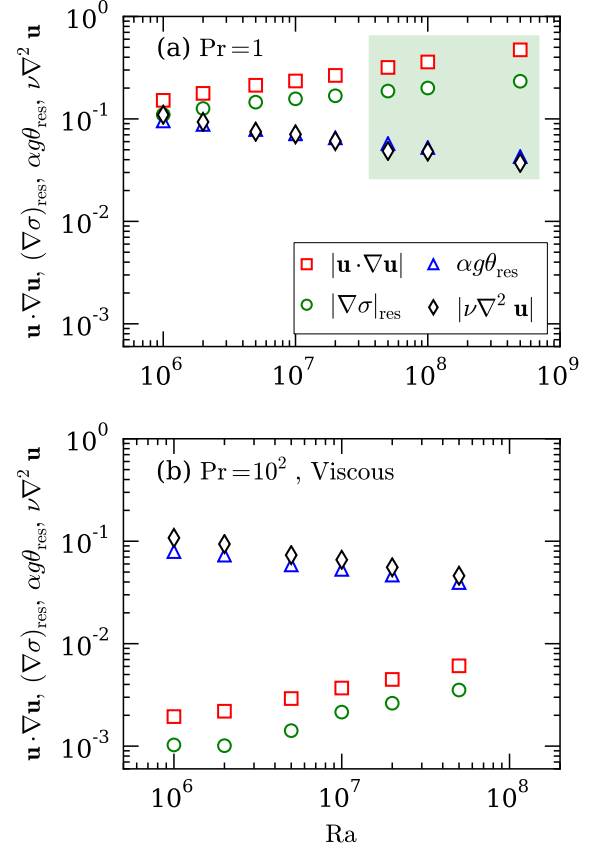


FIG. 3. Plots comparing the root mean square (rms) values of various terms of the momentum equation,  $|\mathbf{u} \cdot \nabla\mathbf{u}|$ ,  $|(\nabla\sigma)_{\text{res}}|$ ,  $\alpha g\theta_{\text{res}}$ , and  $|\nu\nabla^2\mathbf{u}|$  as function of Ra for (a)  $\text{Pr} = 1$ , and (b)  $\text{Pr} = 10^2$ . In the turbulent regime (the shaded region of Fig. (a)),  $|\mathbf{u} \cdot \nabla\mathbf{u}| \approx |\nabla\sigma| \gg \alpha g\theta_{\text{res}}, |\nu\nabla^2\mathbf{u}|$ ; in the viscous regime (Fig. (b)),  $\alpha g\theta_{\text{res}}, |\nu\nabla^2\mathbf{u}| \gg |\mathbf{u} \cdot \nabla\mathbf{u}|, |\nabla\sigma|$ .

and  $256^3$  grids, and find that the Nusselt and Péclet numbers are different by approximately 3% and 1%, respectively.

We compute  $U = \sqrt{\langle u^2 \rangle}$  (rms value) first by volume averaging  $u^2$  over the entire computational domain, and then by performing temporal average over many snapshots (see Table I). We also estimate the strengths of various terms of the momentum equation by similar averaging process. These values are depicted in Fig. 3(a,b) for  $\text{Pr} = 1$  and  $10^2$  with Ra ranging from  $10^6$  to  $5 \times 10^8$ . The flow is turbulent for  $\text{Pr} = 1$  and  $\text{Ra} = 5 \times 10^7, 10^8, 5 \times 10^8$  whose respective Reynolds numbers are approximately 1103, 1537, and 3408. But the flow is laminar for all Ra's when  $\text{Pr} = 10^2$ . Clearly, the numerical values are consistent with the schematic diagram of Fig. 2. The acceleration in the turbulent regime is dominated by the pressure gradient, with small contributions from the buoyancy and viscous terms. However, in the viscous regime, the viscous term balances the buoyancy yielding a very small acceleration. Therefore, all the terms of the momen-

TABLE I. Details of our numerical simulations performed in a unit box: the Prandtl number (Pr), the Rayleigh number (Ra), grid points ( $N^3$ ), the Nusselt number (Nu), the Péclet number (Pe), averaged values of the nonlinear term, pressure gradient, buoyancy, and viscous force, and the number of snapshots over which the time-averaging has been performed. The terms of the momentum equation are normalized by  $Z = \alpha g \Delta$ .

Pr	Ra	$N^3$	Nu	Pe	$ \mathbf{u} \cdot \nabla \mathbf{u} /Z$	$ (-\nabla \sigma)_{\text{res}} /Z$	$ \alpha g \theta_{\text{res}} \hat{z} /Z$	$ \nu \nabla^2 \mathbf{u} /Z$	snapshots
1	$1 \times 10^6$	$60^3$	8.0	146.1	0.152	0.110	0.0955	0.110	100
1	$2 \times 10^6$	$60^3$	10.0	211.3	0.177	0.126	0.0883	0.0934	200
1	$5 \times 10^6$	$60^3$	13.4	340.3	0.213	0.146	0.0788	0.0749	200
1	$1 \times 10^7$	$80^3$	16.3	485.4	0.234	0.157	0.0715	0.0707	50
1	$2 \times 10^7$	$80^3$	20.2	687.4	0.266	0.168	0.0653	0.0608	200
1	$5 \times 10^7$	$80^3$	26.8	1103	0.318	0.187	0.0575	0.0486	178
1	$1 \times 10^8$	$100^3$	32.9	1554	0.359	0.200	0.0526	0.0480	100
1	$1 \times 10^8$	$256^3$	31.9	1537	0.348	0.205	0.0529	0.0789	35
1	$5 \times 10^8$	$256^3$	51.2	3408	0.472	0.233	0.0429	0.0559	34
6.8	$1 \times 10^6$	$60^3$	8.4	182.7	0.0273	0.0160	0.0825	0.106	200
6.8	$2 \times 10^6$	$60^3$	9.9	252.8	0.0333	0.0221	0.0745	0.0865	200
6.8	$5 \times 10^6$	$60^3$	13.1	413.6	0.0413	0.0272	0.0645	0.0691	250
6.8	$1 \times 10^7$	$80^3$	16.2	608.6	0.0474	0.0310	0.0581	0.0669	163
6.8	$2 \times 10^7$	$80^3$	20.3	903.2	0.0558	0.0352	0.0518	0.0570	200
6.8	$5 \times 10^7$	$80^3$	27.7	1536	0.0696	0.0419	0.0452	0.0456	165
$10^2$	$1 \times 10^6$	$60^3$	8.5	190.7	$1.94 \times 10^{-3}$	$1.02 \times 10^{-3}$	0.0792	0.108	300
$10^2$	$2 \times 10^6$	$60^3$	11.2	278.2	$2.19 \times 10^{-3}$	$1.01 \times 10^{-3}$	0.0731	0.0938	180
$10^2$	$5 \times 10^6$	$60^3$	14.5	500.0	$2.91 \times 10^{-3}$	$1.42 \times 10^{-3}$	0.0587	0.0732	250
$10^2$	$1 \times 10^7$	$80^3$	17.1	704.2	$3.69 \times 10^{-3}$	$2.15 \times 10^{-3}$	0.0531	0.0659	250
$10^2$	$2 \times 10^7$	$80^3$	20.7	1044	$4.47 \times 10^{-3}$	$2.62 \times 10^{-3}$	0.0467	0.0556	200
$10^2$	$5 \times 10^7$	$80^3$	27.7	1826	$6.08 \times 10^{-3}$	$3.53 \times 10^{-3}$	0.0395	0.0460	250

tum equation balance reasonably well. To test this out, we compute the vector sum of all the terms for a given Ra, i.e.,  $S = |\mathbf{u} \cdot \nabla \mathbf{u}| - |(-\nabla \sigma)_{\text{res}}| - |\alpha g \theta_{\text{res}} \hat{z}| + |\nu \nabla^2 \mathbf{u}|$ , and the most dominant term among them, i.e.,  $M = \text{Maximum}(|\mathbf{u} \cdot \nabla \mathbf{u}|, |(-\nabla \sigma)_{\text{res}}|, |\alpha g \theta_{\text{res}} \hat{z}|, |\nu \nabla^2 \mathbf{u}|)$ . We observe that the ratio  $S/M$  varies from 19% to 53%. We also remark that the buoyancy in RBC is  $\alpha g(T - T_{\text{ref}})\hat{z}$ . The temperature  $T$  of a cold fluid parcel (moving downward) is lower than that of the reference fluid temperature  $T_{\text{ref}}$ . Therefore the force due to the density difference on a colder fluid parcel traveling downwards is in  $-\hat{z}$  direction, and hence the forces on it are opposite to that in Fig. 2. Thus, the net force balance holds for both ascending and descending fluid parcels.

### III. RESULTS

To quantify the terms of the momentum equation, we perform a scaling analysis of the momentum equation that yields

$$c_1 \frac{U^2}{d} = c_2 \frac{U^2}{d} + c_3 \alpha g \Delta - c_4 \nu \frac{U}{d^2}, \quad (2)$$

where  $c_1 = |\mathbf{u} \cdot \nabla \mathbf{u}|/(U^2/d)$ ,  $c_2 = |(\nabla \sigma)_{\text{res}}|/(U^2/d)$ ,  $c_3 = \theta_{\text{res}}/\Delta$ , and  $c_4 = |\nabla^2 \mathbf{u}|/(U/d^2)$  are dimensionless coefficients. In this paper, we compute the coefficients  $c_i$ 's for  $\text{Pr} = 1, 6.8, 100$ , and  $1000$ . Our numerical com-

putation yields

$$c_1 = 1.5 \text{Ra}^{0.10} \text{Pr}^{-0.06}, \quad (3a)$$

$$c_2 = 1.6 \text{Ra}^{0.09} \text{Pr}^{-0.08}, \quad (3b)$$

$$c_3 = 0.75 \text{Ra}^{-0.15} \text{Pr}^{-0.05}, \quad (3c)$$

$$c_4 = 20 \text{Ra}^{0.24} \text{Pr}^{-0.08}. \quad (3d)$$

The errors in the exponents are  $\lesssim 0.01$ , whereas the prefactors are uncertain by approximately 10%. However in the  $c_4$  vs. Ra scaling, the exponent 0.24 is uncertain by approximately 30%. Clearly the coefficients are weak functions of Pr (see Fig. 4), but they show significant variations with Ra. This aspect is in contrast to *unbounded flows* (without walls) where the coefficients are independent of parameters. We attribute the above scaling to the thermal plates or *bounded flow*. Note that the Ra dependence of  $c_4$  leads to an enhanced viscous force for RBC compared to free turbulence. In our simulations,  $1 \leq \text{Pr} \leq 10^2$  and  $\text{Ra} \leq 5 \times 10^8$ , hence  $c_i$ 's of Eqs. (3a-3d) may get altered for larger Ra and extreme Pr. Also,  $c_i$ 's should depend, at least weakly, on geometry and aspect ratio. Yet we believe that our formula, to be described below, should provide approximate description of  $U$  for large Ra and extreme Pr's.

Multiplication of Eq. (2) with  $d^3/\kappa^2$  yields the following equation for the Péclet number:

$$c_1 \text{Pe}^2 = c_2 \text{Pe}^2 + c_3 \text{RaPr} - c_4 \text{PePr}, \quad (4)$$

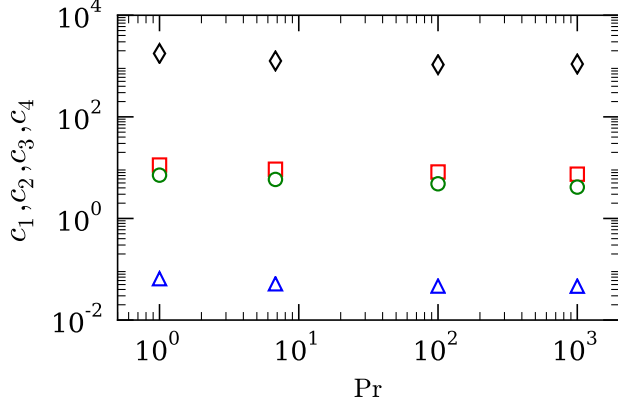


FIG. 4. Plot of  $c_1$  (red squares),  $c_2$  (green circles),  $c_3$  (blue triangles), and  $c_4$  (black diamonds) of Eqs. (3a-3d) as function of  $Pr$  for  $Ra = 2 \times 10^7$ . All  $c_i$ 's are weak functions of  $Pr$ .

whose solution is

$$Pe = \frac{-c_4 Pr + \sqrt{c_4^2 Pr^2 + 4c_3(c_1 - c_2)RaPr}}{2(c_1 - c_2)}. \quad (5)$$

A combination of Eqs. (5, 3a-3d) provide us a predictive formula for the Péclet number for arbitrary  $Pr$  and  $Ra$ . We test our formula with numerical results of ours, Reeuwijk *et al.* [23], Silano *et al.* [24], and Scheel and Schumacher [34], and the experimental results of Niemela *et al.* [18], Xin and Xia [14] and Cioni *et al.* [15]. The predictions of Eq. (5) for  $Pr = 0.022$  and  $Pr = 6.8$  have been multiplied with 2.5 and 1.2, respectively, to fit the experimental results from Cioni *et al.* [15] and Xin and Xia [14]. As shown in Fig. 5, our formula describes the numerical and experimental data reasonably well for Prandtl numbers ranging from 0.025 to 1000 and for various geometries. However the above factors (2.5 and 1.2) and the discrepancy between our predictions and the results of Niemela *et al.* [18] are due to the aforementioned uncertainty in  $c_i$ 's, geometrical factors, aspect ratio dependence, and different definitions used for  $U$ .

The limiting cases of  $Pe$  formula of Eq. (5) along with Eqs. (3a-3d) yield *turbulent* and *viscous* regimes. For  $c_4^2 Pr^2 \ll 4c_3(c_1 - c_2)RaPr$  or  $Ra \gg 10^6 Pr$ , we obtain the turbulent regime with

$$Pe = \sqrt{\frac{c_3}{c_1 - c_2}} RaPr \approx \sqrt{7.5 Pr} Ra^{0.38}. \quad (6)$$

For mercury ( $Pr = 0.025$ ), the above formula yields  $(Pe)_{\text{theory}} = 0.38 Ra^{0.38}$ , which is in a reasonable agreement with Cioni *et al.*'s [15] finding that  $(Pe)_{\text{expt}} = 0.24 Ra^{0.43}$  in their convection experiment with mercury. However, for  $Pr \approx 1$  and large  $Ra$ , many authors [13, 14, 16, 18, 22, 23, 26, 27, 34] as well as our numerical simulation show that  $Pe \sim Ra^{0.51}$ , not  $Pe \sim Ra^{0.38}$ .

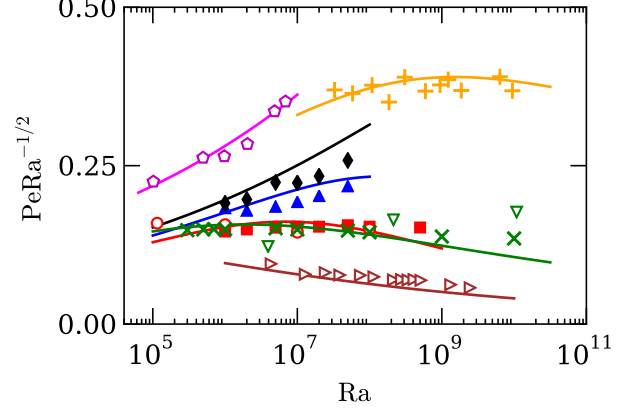


FIG. 5. Plot of normalized Péclet number  $PeRa^{-1/2}$  as a function of  $Ra$  for different  $Pr$ . Our formula for Péclet number  $Pe$  [Eq. (5)] (shown as continuous curves) fits reasonably well with the numerical results of ours ( $Pr = 1$ , red squares;  $Pr = 6.8$ , blue triangles;  $Pr = 10^2$ , black diamonds), Scheel and Schumacher [34] ( $Pr = 0.7$ , green crosses), Reeuwijk *et al.* [23] ( $Pr = 1$ , red circles), and Silano *et al.* [24] ( $Pr = 10^3$ , magenta pentagons), and the experimental results of Cioni *et al.* [15] ( $Pr \approx 0.022$ , brown right-triangles) and Xin and Xia [14] ( $Pr \approx 6.8$ , orange pluses). The agreement is deficient for Niemela *et al.*'s [18] experimental result ( $Pr \approx 0.7$ , green down-triangles), possibly due to various factors discussed in the text.

This is because the condition  $Ra \gg 10^6 Pr$  is not satisfied in these systems (except for [18] and [34]). We remark that Eqs. (3a-3d) and the condition  $Ra \gg 10^6 Pr$  are derived from our numerical simulations for moderate values of  $Ra$  and  $Pr$ ; these equations may change somewhat for larger Rayleigh and Prandtl numbers. Aspect ratio and geometry could also affect the scaling of  $c_i$ 's. Yet, our  $Pe$  formula describes the earlier experimental and numerical simulations reasonably well (see Fig. 5).

The other limiting case (viscous regime),  $c_4^2 Pr^2 \gg 4c_3(c_1 - c_2)RaPr$ , yields  $Pe = (c_3/c_4)Ra \approx 0.0375 Ra^{0.61}$ , which is independent of  $Pr$  as observed in several numerical simulations and experiments [13, 19, 24, 29, 30]. An application of the above to the Earth's mantle [35–37] with parameters  $d \approx 2900$  km,  $\kappa \approx 10^{-6} \text{m}^2/\text{s}$ ,  $Ra \approx 5 \times 10^7$ ,  $Pr \approx 10^{23} - 10^{24}$ , and  $U \approx 20$  mm/yr yields  $(Pe)_{\text{obs}} \approx 1840$ , which is close to our prediction that  $(Pe)_{\text{theory}} \approx 1580$ .

The effects of the thermal plates and the boundary layers come into play in several different ways in our analysis. Firstly, the thermal boundary layers at the two walls induce a mean temperature profile  $\theta_m(z)$  (see Fig. 1), and only  $\theta_{\text{res}} = \theta - \theta_m(z)$  participate in the momentum equation. Secondly, the boundary layers induce  $Ra$  dependence on  $c_i$ 's [see Eq. (3a-3d)]. Note that in unbounded flows (without walls) the corresponding  $c_i$ 's are expected to be constants, i.e., independent of system parameters like  $Ra$ . Also, the ratio of the nonlinear term and the vis-

TABLE II. Ra dependence of the normalized correlation function  $C_{u\theta_{\text{res}}} = \langle u_z \theta_{\text{res}} \rangle / [\langle u_z^2 \rangle^{1/2} \langle \theta_{\text{res}}^2 \rangle^{1/2}]$ ,  $\langle \theta_{\text{res}}^2 \rangle^{1/2}$ ,  $\langle u_z^2 \rangle^{1/2}$ , Nu, and  $\epsilon_u$  computed using numerical data. The data in the shaded region of Fig. 3(a) for Pr = 1 belongs to the turbulent regime whereas that of Pr = 10<sup>2</sup> belongs to the viscous regime. The functional forms of the table are based on our numerical simulations. The errors in the exponents are around 5%–10% for all the quantities.

	Turbulent regime	Viscous regime
$C_{u\theta_{\text{res}}}$	$\text{Ra}^{-0.05}$	$\text{Ra}^{-0.07}$
$\langle \theta_{\text{res}}^2 \rangle^{1/2}$	$\text{Ra}^{-0.13}$	$\text{Ra}^{-0.18}$
$\langle u_z^2 \rangle^{1/2}$	$\text{Ra}^{0.51}$	$\text{Ra}^{0.58}$
Nu	$\text{Ra}^{0.32}$	$\text{Ra}^{0.33}$
$\epsilon_u$	$(U^3/d)\text{Ra}^{-0.21}$	$(\nu U^2/d^2)\text{Ra}^{0.17}$

cous term of Eq. (2) is  $(Ud/\nu)(c_1/c_4) \sim \text{ReRa}^{-0.14}$ , not just Reynolds number Re as in unbounded flows. Thus, the thermal plates or the boundary layers enhance the dissipation in RBC flows compared to unbounded flows. We show below that a similar behaviour is observed for the Nusselt number and the viscous dissipation rates.

The anisotropy induced by the thermal plates have important consequences on the heat transport and dissipation rates:  $\langle u_z \theta \rangle \neq 0$ , unlike  $\langle u_z \theta \rangle = 0$  for any isotropic flow. The Nusselt number  $\text{Nu} \sim C_{u\theta_{\text{res}}} \langle u_z^2 \rangle^{1/2} \langle \theta_{\text{res}}^2 \rangle^{1/2}$ , where  $C_{u\theta_{\text{res}}} = \langle u_z \theta_{\text{res}} \rangle / [\langle u_z^2 \rangle^{1/2} \langle \theta_{\text{res}}^2 \rangle^{1/2}]$  is the normalized correlation function. In Table II we list the Ra dependence for the above quantities that provide the appropriate corrections to the Nusselt number from the Kraichnan's predictions [38] for the ultimate regime ( $\text{Nu} \sim \text{Ra}^{1/2}$ ) to the experimentally observed  $\text{Nu} \sim \text{Ra}^{0.30}$ .

It is tempting to connect our findings to the ultimate regime of turbulent convection [20, 21]. We conjecture that in the ultimate regime,  $C_{u\theta_{\text{res}}}$  and  $\theta_{\text{res}}$ , as well as the coefficients  $c_i$ 's, would become independent of Ra due to boundary layer detachment, and hence yield  $\text{Nu} \sim \text{Ra}^{1/2}$ . We need inputs from experiments and numerical simulations to probe the above conjectures.

The exact relations of Shraiman and Siggia [39] yields

$$\epsilon_u = \frac{U^3}{d} \frac{(\text{Nu} - 1)\text{RaPr}}{\text{Pe}^3}. \quad (7)$$

In the turbulent regime,  $\text{Nu} \sim \text{Ra}^{0.32}$  and  $\text{Pe} \sim \text{Ra}^{0.51}$ , hence,  $\epsilon_u \neq U^3/d$ , rather  $\epsilon_u \sim (U^3/d)\text{Ra}^{-0.21}$  due the confinement of turbulence by the walls. In the viscous regime,

$$\epsilon_u = \frac{\nu U^2}{d^2} \frac{(\text{Nu} - 1)\text{Ra}}{\text{Pe}^2}. \quad (8)$$

Since  $\text{Nu} \sim \text{Ra}^{0.33}$  and  $\text{Pe} \sim \text{Ra}^{0.58}$  [24, 29], we observe that  $\epsilon_u \sim (\nu U^2/d^2)\text{Ra}^{0.17}$ . Note that in a typical viscous scenario,  $\epsilon_u \sim \nu U^2/d^2$ . Hence the Ra dependence of  $c_4$  and the correction of the viscous term from  $\nu U/d^2$  are related to the boundary layers.

To quantify the scaling of the viscous dissipation rates given by Eqs. (7, 8), we define two normalized viscous

dissipation rates as

$$C_{\epsilon_u,1} = \frac{\epsilon_u}{U^3/d} = \frac{(\text{Nu} - 1)\text{RaPr}}{\text{Pe}^3} \sim \text{Ra}^{-0.21}\text{Pr}, \quad (9)$$

$$C_{\epsilon_u,2} = \frac{\epsilon_u}{\nu U^2/d^2} = \frac{(\text{Nu} - 1)\text{Ra}}{\text{Pe}^2} \sim \text{Ra}^{0.17}. \quad (10)$$

The correlation functions  $C_{\epsilon_u,1}, C_{\epsilon_u,2}$  are suitable for the turbulent and viscous regimes respectively. We compute these quantities using the numerical data obtained from numerical simulations and plot them as a function of Ra in Fig. 6, and demonstrate that  $C_{\epsilon_u,1} \sim \text{Ra}^{-0.21}$  and  $C_{\epsilon_u,2} \sim \text{Ra}^{0.17}$ . Our results demonstrate that the dissipation rates in RBC differ from those in unbounded flows due to the walls.

Let us estimate the ratio of the total dissipation rates (product of the dissipation rate and the appropriate volume) in the boundary layer ( $D_{u,BL}$ ) and in the bulk ( $D_{u,bulk}$ ). In the turbulent regime

$$\begin{aligned} \frac{D_{u,BL}}{D_{u,bulk}} &\approx \frac{(\epsilon_{u,BL})(2A\delta_u)}{(\epsilon_{u,bulk})(Ad - 2A\delta_u)} \approx \left( \frac{2\nu U^2/\delta_u^2}{U^3/d} \right) \frac{\delta_u}{d} \\ &\approx 2 \frac{d/\delta_u}{\text{Re}} \approx 2\text{Re}^{-1/2} \end{aligned} \quad (11)$$

since  $\delta_u/d \sim \text{Re}^{-1/2}$  [40]. Here  $\delta_u$  is the thickness of the viscous boundary layers at the top and bottom plates, and  $A$  is the cross section area of these plates. The factor 2 is included to account for the dissipation near both the plates. Equation (11) indicates that  $D_{u,BL} \ll D_{u,bulk}$  for large Re. In the viscous regime, the boundary layer spans the whole region ( $2\delta_u \approx d$ ), therefore  $D_{u,BL}$  dominates  $D_{u,bulk}$ . Also, our formula [Eq. (5)] includes both the turbulent and viscous regimes.

Earlier Grossmann and Lohse [10–12] estimated  $U$  and the Nusselt number by invoking the exact relations of Shraiman and Siggia [39] and using the fact that the total dissipation is a sum of those in the bulk and in the boundary layers ( $D_{u,bulk}$  and  $D_{u,BL}$  respectively). Our derivation is an alternative to that of GL with an attempt to highlight the anisotropic effects arising due to the boundary layers that yield  $\epsilon_u \neq U^3/d$  and  $\epsilon_T \neq U\Delta^2/d$ . Note that our derivation does not use Shraiman and Siggia's [39] exact relations,  $\epsilon_{u,bulk} \sim U^3/d$ , and  $\epsilon_{T,bulk} \sim U\Delta^2/d$ , where  $\epsilon_{u,bulk}$  and  $\epsilon_{T,bulk}$  are the bulk viscous and thermal dissipation rates respectively.

#### IV. CONCLUSIONS

The agreement between our model and earlier experiments and numerical simulations is remarkable, considering that our prediction is based on cubical box, while the experiments and numerical simulations employ cubical and cylindrical geometry. This result indicates that the Péclet scaling is *weakly* dependent on the aspect ratio or geometry, and it can be reasonably well described by the nonlinear equation [Eq. (2)] which is based on the scaling of the large-scale quantities. This is one of the important



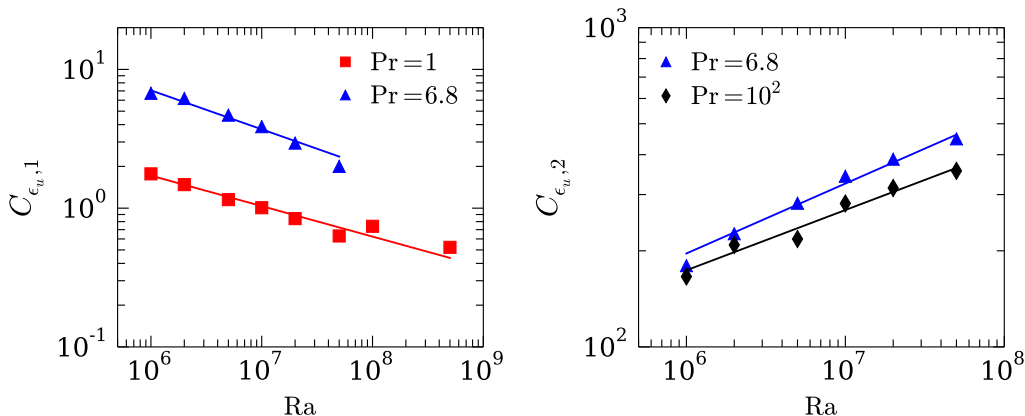


FIG. 6. The plots of normalized viscous dissipation rates  $C_{\epsilon_u,1}$  (in turbulent regime) and  $C_{\epsilon_u,2}$  (in viscous regime).  $C_{\epsilon_u,1} \sim Ra^{-0.22}$  and  $Ra^{-0.25}$  for  $Pr = 1$  and  $6.8$  respectively, while  $C_{\epsilon_u,2} \sim Ra^{0.22}$  and  $Ra^{0.19}$  for  $Pr = 6.8$  and  $10^2$  respectively. These results are in qualitative agreement with model predictions (see Table II).

conclusions of our work as well as that of Grossmann and Lohse [10–12]. Note however that the discrepancies between the model predictions and the experiments results (see Fig. 5) could be due to weak dependence of Péclet number on geometry or aspect ratio.

The above observations indicate that the flow behaviour in RBC differs significantly from the unbounded hydrodynamic turbulence for which we employ homogeneous and isotropic formalism. Interestingly, in turbulent RBC, the buoyancy term is *nearly* cancelled by the viscous term. This feature of RBC could be the reason for the Kolmogorov’s spectrum in RBC, as reported by Kumar *et al.* [41]. The aforementioned wall effects should also be present in other bounded flows such as in channels, pipes, rotating convection, spheres, and cylinders. The procedure adopted in this paper would yield similar formulae for the large-scale velocity and the dissipation rate for these systems.

In summary, we derive a general formula for the large scale velocity  $U$  or the Péclet number for RBC that is applicable for any  $Ra$  and  $Pr$ . Our formula provides reasonable fits to the results of earlier experiments and numerical simulations. We also compute the correlation function between  $u_z$  and  $\theta$  that causes deviations of the Nusselt number from the theoretical prediction of  $Nu \sim Ra^{1/2}$  to the experimentally observed  $Nu \sim Ra^{0.30}$ . In Table II, we also show how the dissipation rate  $\epsilon_u$  and the temperature fluctuations in RBC get corrections from the usual formulas due to the boundary walls.

Our formulae discussed in this paper provide insights into the flow dynamics of RBC. These results will be useful in modelling convective flows in the interiors and atmospheres of stars and planets, as well as in engineering applications.

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## APPENDIX: TEMPERATURE PROFILE AND BOUNDARY LAYER

The temperature is a function of  $x, y, z$ , i.e.,  $T(x, y, z)$ . However, as shown in Fig. 1, its planar average,  $T_m(z)$ , is approximately  $1/2$  in the bulk, and it varies rapidly in the boundary layers. We can approximate  $T_m(z)$  as

$$T_m(z) = \begin{cases} 1 - \frac{z}{2\delta_T} & \text{for } 0 < z < \delta_T \\ 1/2 & \text{for } \delta_T < z < 1 - \delta_T \\ \frac{1-z}{2\delta_T} & \text{for } 1 - \delta_T < z < 1 \end{cases} \quad (12)$$

where  $\delta_T$  is the thickness of the thermal boundary layers at the top and bottom plates. In RBC, it is customary to describe the flow using the temperature fluctuation from the conduction state,  $\theta$ , defined as

$$T(x, y, z) = \theta(x, y, z) + 1 - z. \quad (13)$$

For the above, we have normalized the temperature fluctuation using the temperature difference between the plates, and the vertical coordinate using the vertical distance between the plates. When we perform averaging of Eq. (13) over  $xy$  planes, we obtain

$$\theta_m(z) = T_m(z) + z - 1, \quad (14)$$

where  $\theta_m(z)$  is

$$\theta_m(z) = \begin{cases} z \left(1 - \frac{1}{2\delta_T}\right) & \text{for } 0 < z < \delta_T \\ z - 1/2 & \text{for } \delta_T < z < 1 - \delta_T \\ (z - 1) \left(1 - \frac{1}{2\delta_T}\right) & \text{for } 1 - \delta_T < z < 1 \end{cases} \quad (15)$$

which is exhibited in Fig. 1. For a pair of thin boundary layers, the Fourier transform of  $\theta_m(z)$ ,  $\hat{\theta}_m(k_z)$ , is dominated by the contributions from the bulk, that is,

$$\begin{aligned} \hat{\theta}_m(k_z) &= \int_0^1 \theta_m(z) \sin(k_z \pi z) dz \\ &\approx \int_0^1 (z - 1/2) \sin(k_z \pi z) dz \\ &\approx \begin{cases} -\frac{1}{\pi k_z} & \text{for even } k_z \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (16)$$

It is interesting to note that the corresponding velocity mode,  $\hat{u}_z(0, 0, k_z) = 0$  because of the incompressibility condition  $\mathbf{k} \cdot \hat{\mathbf{u}}(0, 0, k_z) = k_z \hat{u}_z(0, 0, k_z) = 0$ . Also,  $\hat{u}_{x,y}(0, 0, k_z) = 0$  when we assume an absence of a mean

horizontal flow in any horizontal plane. Hence, the momentum equation for the Fourier mode  $(0, 0, k_z)$  is

$$-ik_z \hat{\sigma}_m(0, 0, k_z) + \alpha g \hat{\theta}_m(0, 0, k_z) = 0, \quad (17)$$

and it does not involve the velocity field. In the real space, the above equation translates to  $d\sigma_m/dz = \alpha g \theta_m(z)$ . For the Fourier modes other than  $(0, 0, k_z)$ , the momentum equation is

$$\begin{aligned} \frac{\partial \hat{\mathbf{u}}(\mathbf{k})}{\partial t} &= -i \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} [\mathbf{k} \cdot \hat{\mathbf{u}}(\mathbf{q})] \hat{\mathbf{u}}(\mathbf{p}) - i \mathbf{k} \hat{\sigma}_{\text{res}}(\mathbf{k}) \\ &\quad + \alpha g \hat{\theta}_{\text{res}}(\mathbf{k}) \hat{\mathbf{z}} - \nu k^2 \hat{\mathbf{u}}(\mathbf{k}). \end{aligned} \quad (18)$$

We denote the participating temperature field in the above equation as residual temperature  $\theta_{\text{res}}$ , and the residual pressure field as  $\sigma_{\text{res}}$ , and they are defined as

$$\theta_{\text{res}} = \theta - \theta_m, \quad (19)$$

$$\sigma_{\text{res}} = \sigma - \sigma_m. \quad (20)$$

Thus, the large-scale velocity,  $U$ , depends on  $\theta_{\text{res}}$  and  $\sigma_{\text{res}}$ .

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